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Cosmological solutions in the brane–bulk system

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Abstract. In this work we find cosmological solutions in the brane–bulk system starting from a 5D line element which is a simple extension, for cosmological applications, of the pioneering Randall–Sundrum line element. From the knowledge of the bulk metric, assumed to have the form of plane waves propagating in the fifth dimension, we solve the corresponding 4D Einstein equations on the brane with a well defined energy-momentum tensor.

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1 Introduction

Brane cosmologies are often studied in the framework of five-dimensional (5D) Einstein equations in the bulk. The effective 4D gravitational equations in the brane without curvature correction terms were first obtained by Shiromizu, Maeda and Sasaki [1]. However, even taking more generalized gravitational actions, the derived 4D Einstein equations do not in general form a closed system due to the presence of a Weyl term which can only be specified in terms of the bulk metric, so other equations are to be written down and different procedures arise in splitting the non-Einsteinian terms between bulk and brane [2]. Because the specific form of a solution in the bulk is in general rather cumbersome due to the number of terms and parameters in the equations, we shall consider, as a guide for further work, a model simple enough to allow obtaining not trivial exact solutions. This paper is organised as follows. In the next section we summarize the effective brane equations obtained by Kofinas [3] when the intrinsic curvature scalar ${}^{(4)}R$ is included in the brane action. In Sect. 3 we transform the static line element proposed by Randall and Sundrum [4] into a dynamical one containing a three-space with constant curvature and obtain, using the 5D Einstein equations, the corresponding energy-momentum tensor in the bulk. In Sect. 4 we find the related bulk metric in coordinate systems commonly used in cosmological applications. Solutions in the brane, assumed infinitely thin and $\mathbf{Z_2}$ symmetric in the bulk, are found in Sect. 5. Finally, conclusions are given in Sect. 6.

2 Braneworld Einstein field equations

In this section we recall the effective brane equations obtained by Kofinas [3], which we shall use in the following, giving a brief account of their derivation. Once we have solved the equations in the bulk, the form of the induced equations will allow us finding brane solutions following the methods of general relativity with a well defined energy-momentum tensor. The starting point in [3] is a three-dimensional brane Σ embedded in a five-dimensional spacetime M. For convenience a coordinate y is chosen such that the hypersurface $y = 0$ coincides with the brane. The total action for the system is taken to be

$$\begin{aligned}\mathcal{S} = &\frac{1}{2\,\kappa_5^2}\int_M \sqrt{-{}^{(5)}g}\,\left({}^{(5)}R - 2\Lambda_5\right)\,\mathrm{d}^5x \\ &+\frac{1}{2\,\kappa_4^2}\int_\Sigma \sqrt{-{}^{(4)}g}\,\left({}^{(4)}R - 2\Lambda_4\right)\,\mathrm{d}^4x \\ &+\int_M \sqrt{-{}^{(5)}g}\,L_5^{\mathrm{mat}}\,\mathrm{d}^5x + \int_\Sigma \sqrt{-{}^{(4)}g}\,L_4^{\mathrm{mat}}\,\mathrm{d}^4x.\end{aligned} \qquad (1)$$

Bulk indices will be denoted by capital Latin letters and brane indices by lower Greek letters. Varying (1) with respect to the bulk metric g_{AB} one obtains the equations

$${}^{(5)}G_A^B = -\Lambda_5\delta_A^B + \kappa_5^2\left({}^{(5)}T_A^B + {}^{(\mathrm{loc})}T_A^B\delta(y)\right), \qquad (2)$$

where

$${}^{(\mathrm{loc})}T_A^B = -\frac{1}{\kappa_4^2}\sqrt{\frac{-{}^{(4)}g}{-{}^{(5)}g}}\,\left({}^{(4)}G_A^B - \kappa_4^2\,{}^{(4)}T_A^B + \Lambda_4 h_A^B\right) \qquad (3)$$

is the localized energy-momentum tensor of the brane. ${}^{(5)}G_{AB}$ and ${}^{(4)}G_{AB}$ denote the Einstein tensors constructed from the bulk and the brane metrics respectively, while the tensor $h_{AB} = g_{AB} - n_A n_B$ is the induced metric on the hypersurfaces y = constant, with n^A the normal unit vector on these:

$$n^A = \frac{\delta_5^A}{\Phi}, \qquad n_A = (0, 0, 0, 0, \Phi). \qquad (4)$$

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The way the coordinate y has been chosen allows one to write the five-dimensional line element, at least in the neighborood of the brane, as

$$\mathrm{d}S^2 = g_{AB}\,\mathrm{d}x^A\mathrm{d}x^B = g_{\mu\nu}\,\mathrm{d}x^\mu\mathrm{d}x^\nu + \Phi^2\mathrm{d}y^2. \qquad (5)$$

Using the methods of canonical analysis [5] the Einstein equations (2) in the bulk are split into the following sets of equations:

$$K^\nu_{\mu;\nu} - K_{;\mu} = \kappa_5^2\,\Phi\,{}^{(5)}T^y_\mu, \qquad (6a)$$

$$K^\mu_\nu K^\nu_\mu - K^2 + {}^{(4)}R = 2\,(\Lambda_5 - \kappa_5^2\,{}^{(5)}T^y_y), \qquad (6b)$$

$$\begin{aligned}&\frac{\partial K^\mu_\nu}{\partial y} + \Phi\,K\,K^\mu_\nu - \Phi\,{}^{(4)}R^\mu_\nu + g^{\mu\lambda}\,\Phi_{;\lambda\nu}\\ &= -\kappa_5^2\,\Phi\left({}^{(\mathrm{loc})}T^\mu_\nu - \frac{1}{3}\,{}^{(\mathrm{loc})}T\,\delta^\mu_\nu\right)\delta(y) - \kappa_5^2\,\Phi\,{}^{(5)}T^\mu_\nu\\ &\quad -\kappa_5^2\,\Phi\,{}^{(5)}T^\mu_\nu + \frac{1}{3}\,\Phi\,(\kappa_5^2\,{}^{(5)}T - 2\Lambda_5)\,\delta^\mu_\nu, \end{aligned} \qquad (6c)$$

where $K_{\mu\nu}$ is the extrinsic curvature of the hypersurfaces $y =$ constant:

$$K_{\mu\nu} = \frac{1}{2\,\Phi}\,\frac{\partial g_{\mu\nu}}{\partial y}, \quad K_{Ay} = 0. \qquad (7)$$

The Israel conditions [6] for the singular part in (6c) are

$$[K^\mu_\nu] = -\kappa_5^2\,\Phi_0\left({}^{(\mathrm{loc})}T^\mu_\nu - \frac{1}{3}\,{}^{(\mathrm{loc})}T\,\delta^\mu_\nu\right), \qquad (8)$$

where the bracket means discontinuity of the quantity across $y = 0$ and $\Phi_0 = \Phi(y = 0)$. Hereafter, considering a $\mathbf{Z_2}$ symmetry on reflection around the brane, (3) become

$${}^{(4)}G^\mu_\nu = -\Lambda_4\,\delta^\mu_\nu + \kappa_4^2\,{}^{(4)}T^\mu_\nu + \frac{2}{r_c}\,(\overline{K}^\mu_\nu - \overline{K}\delta^\mu_\nu), \qquad (9)$$

where $\overline{K}^\mu_\nu = K^\mu_\nu(y = 0^+) = -K^\mu_\nu(y = 0^-)$ and $r_c = \kappa_5^2/\kappa_4^2$ is a distance scale. Equations (9) assume the form of the usual Einstein equations with the energy-momentum tensor split into the common brane energy-momentum tensor plus additional terms which are all multiplied by $1/r_c$. The tensor ${}^{(4)}T^\mu_\nu$ satisfies the usual conservation law ${}^{(4)}T^\mu_{\nu;\mu} = 0$ provided ${}^{(5)}T^y_\mu = 0$, which means no exchange of energy between brane and bulk. The quantities $\overline{K}^\mu_\nu$ are still undetermined, however additional information can be obtained from the geometrical identity

$$\begin{aligned}{}^{(4)}R^A_{BCD} &= {}^{(5)}R^M_{NKL}\,h^A_M\,h^N_B\,h^K_C\,h^L_D\\ &\quad +(K^A_C\,K_{BD} - K^A_D\,K_{BC}).\end{aligned} \qquad (10)$$

Taking suitable contractions from the above relation it is possible to construct the four- and five-dimensional Einstein tensors and, making use of the bulk Einstein equations, to get finally the parallel to the brane equations

$$\begin{aligned}&{}^{(4)}G^\mu_\nu\\ &= -\frac{1}{2}\,\Lambda_5\delta^\mu_\nu + \frac{2}{3}\,\kappa_5^2\left({}^{(5)}\overline{T}^\mu_\nu + \left({}^{(5)}\overline{T}^y_y - \frac{1}{4}\,{}^{(5)}\overline{T}\right)\delta^\mu_\nu\right)\\ &\quad + \left(\overline{K}\,\overline{K}^\mu_\nu - \overline{K}^\mu_\lambda\,\overline{K}^\lambda_\nu\right) + \frac{1}{2}\left(\overline{K}^\kappa_\lambda\,\overline{K}^\lambda_\kappa - \overline{K^2}\right)\delta^\mu_\nu\\ &\quad -g^{\kappa\mu}\,{}^{(5)}\overline{C}^y_{\kappa y\nu}.\end{aligned} \qquad (11)$$

Here ${}^{(5)}C^y_{\kappa y\nu}$ is the "electric" part of the bulk Weyl tensor, while $\overline{T}$ and $\overline{C}$ are the limiting values of those quantities at $y = 0^+$ or 0^-. Now, equating the right-hand sides of the independent equations (9) and (11), one gets an algebraic equation for $\overline{K}^\mu_\nu$ which can be substituted in (9) once the equation has been solved. However the system of Einstein equations for the brane metric so obtained is not, in general, closed due to the presence of the Weyl term, so one has to solve the Einstein equations in the bulk in order to determine the Weyl tensor on the brane. This one will be the method followed in this paper. A different approach, which instead does not assume a bulk geometry, starts from deducing a brane dynamics and then searches for a bulk geometry in which the brane can consist of its boundary [7].

3 The model

Following a possibility suggested in [8] we shall consider 5D Einstein equations strictly in the bulk, i.e. without the brane energy-momentum tensor with its delta distribution, so when searching, later on, for solutions in the brane we shall have to take limiting values of bulk quantities as discussed in the context of (11). As a simplifying device for dealing with these equations, we select a five-dimensional line element with reasonable physical properties, calculate the corresponding energy-momentum tensor ${}^{(5)}T^B_A$ and, once solved (11) for the brane metric, we can obtain, from (9), the 4D cosmological constant Λ_4 and the effective energy-momentum tensor on the brane. More in detail, let us consider the static Randall–Sundrum line element

$$\mathrm{d}S^2 = \mathrm{e}^{-2\kappa y}\left(A^2\,\mathrm{d}\sigma_k^2 - \mathrm{d}t^2\right) + \mathrm{d}y^2, \qquad (12)$$

containing a three-space with constant curvature. Here κ and A are, respectively, the constant scale factors for the extra dimension y and for the ordinary three-space and $\mathrm{d}\sigma_k^2$ is the line element on maximally symmetric three-spaces with curvature index $k = +1, 0, -1$:

$$\mathrm{d}\sigma_k^2 = \frac{\mathrm{d}r^2}{1 - kr^2} + r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2). \qquad (13)$$

Since our purpose is to describe the time evolution on the braneworld, we need to transform the static bulk solution (12) into a dynamical one. We follow the procedure used in [9–11] where dynamical solutions are derived from the static Randall–Sundrum metric by generalized boosts along the fifth dimension. Applied to the actual case, we write the required transformations as

$$\begin{cases} t = \dfrac{\frac{1-\mathrm{e}^{-\widetilde{\chi}\,\overline{t}}}{\chi} - \frac{\chi}{\kappa^2}\mathrm{e}^{\widetilde{\kappa}\,\overline{y}}}{\sqrt{1-\frac{\chi^2}{\kappa^2}}}, \\ \\ \mathrm{e}^{\kappa\,y} = \dfrac{\mathrm{e}^{\widetilde{\kappa}\,\overline{y}} + \mathrm{e}^{-\widetilde{\chi}\,\overline{t}} - 1}{\sqrt{1-\frac{\chi^2}{\kappa^2}}}, \end{cases} \tag{14}$$

where $\widetilde{\kappa} = \frac{\kappa}{\sqrt{1-\frac{\chi^2}{\kappa^2}}}$, $\widetilde{\chi} = \frac{\chi}{\sqrt{1-\frac{\chi^2}{\kappa^2}}}$ and χ is a constant responsible of the boost in the (t, y) spacetime.

As a result the metric (12) becomes, dropping the bar,

$$\mathrm{d}S^2 = \frac{1}{(\mathrm{e}^{-\widetilde{\chi}\,t} + \mathrm{e}^{\widetilde{\kappa}\,y} - 1)^2} \times \left(\widetilde{A}^2\,\mathrm{d}\sigma_k^2 - \mathrm{e}^{-2\widetilde{\chi}\,t}\,\mathrm{d}t^2 + \mathrm{e}^{2\widetilde{\kappa}\,y}\,\mathrm{d}y^2\right), \tag{15}$$

with $\widetilde{A}^2 = \left(1 - \frac{\chi^2}{\kappa^2}\right) A^2$.

Having specified the metric components and consequently the components of the Einstein tensor, we have

$$G_r^r = G_\vartheta^\vartheta = G_\varphi^\varphi = -6\,(\widetilde{\chi}^2 - \widetilde{\kappa}^2) - \frac{k}{a^2(t,y)}, \tag{16a}$$

$$G_t^t = G_y^y = -6\,(\widetilde{\chi}^2 - \widetilde{\kappa}^2) - \frac{3\,k}{a^2(t,y)}, \tag{16b}$$

where

$$a(t,y) = \frac{\widetilde{A}}{\mathrm{e}^{-\widetilde{\chi}\,t} + \mathrm{e}^{\widetilde{\kappa}\,y} - 1}. \tag{17}$$

By comparison with (16) and taking into account that G_t^r and G_t^y are both equal to zero, we obtain

$$\begin{aligned} &\Lambda_5 = 6\,(\widetilde{\chi}^2 - \widetilde{\kappa}^2) = -6\,\kappa^2, \\ &{}^{(5)}T_A^B = \mathrm{diag}(p, p, p, -\rho, p_\perp), \end{aligned} \tag{18}$$

with

$$\kappa_5^2\,p = -\frac{k}{a^2(t,y)}, \quad \kappa_5^2\,\rho = -\kappa_5^2\,p_\perp = \frac{3\,k}{a^2(t,y)}. \tag{19}$$

It is worth noticing that the bulk sources contain, besides the cosmological term which can be interpreted as proportional to the pressure and to the density of vacuum fluctuations [12], terms proportional to the curvature index k which call in mind, even if k and $a^2(t,y)$ refer to different spaces, a relationship with the "pressure" and "density" coming from the curvature of the universe [13]. The line element (15) is a particular realization of the metric

$$\mathrm{d}S^2 = a^2(t,y)\,\mathrm{d}\sigma_k^2 - n^2(t,y)\,\mathrm{d}t^2 + \Phi^2(t,y)\,\mathrm{d}y^2, \tag{20}$$

commonly used in cosmological applications, so it is worth obtaining the above metric coefficients solving Einstein with a generic cosmological constant Λ_5 and the tensor ${}^{(5)}T_A^B$ whose components are given in (19).

4 Solutions in the bulk

In the coordinate system (20) the non-vanishing components of the Einstein tensor G_A^B are

$$\begin{aligned} G_r^r = G_\vartheta^\vartheta = G_\varphi^\varphi &= -\frac{1}{n^2}\left[\frac{\ddot{\Phi}}{\Phi} + \frac{2\ddot{a}}{a} + \frac{\dot{\Phi}}{\Phi}\left(\frac{2\dot{a}}{a} - \frac{\dot{n}}{n}\right) + \frac{\dot{a}}{a}\left(\frac{\dot{a}}{a} - \frac{2\dot{n}}{n}\right)\right] \\ &+ \frac{1}{\Phi^2}\left[\frac{2a''}{a} + \frac{n''}{n} + \frac{a'}{a}\left(\frac{a'}{a} + \frac{2n'}{n}\right) - \frac{\Phi'}{\Phi}\left(\frac{2a'}{a} + \frac{n'}{n}\right)\right] - \frac{k}{a^2}, \end{aligned} \tag{21a}$$

$$G_t^t = -\frac{3}{n^2}\left(\frac{\dot{a}^2}{a^2} + \frac{\dot{a}\dot{\Phi}}{a\Phi}\right) + \frac{3}{\Phi^2}\left(\frac{a''}{a} + \frac{a'^2}{a^2} - \frac{a'\Phi'}{a\Phi}\right) - \frac{3k}{a^2}, \tag{21b}$$

$$G_y^y = -\frac{3}{n^2}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} - \frac{\dot{a}\dot{n}}{an}\right) + \frac{3}{\Phi^2}\left(\frac{a'^2}{a^2} + \frac{a'n'}{an}\right) - \frac{3k}{a^2}, \tag{21c}$$

$$G_y^t = -\frac{3}{\Phi^2}\left(\frac{\dot{a}'}{a} - \frac{\dot{a}n'}{an} - \frac{a'\dot{\Phi}}{a\Phi}\right). \tag{21d}$$

Here a dot and a prime denote partial derivatives with respect to t and y, respectively. The specific form of the solution in the bulk is in general rather cumbersome, but some simplifications can arise if one makes the assumption of plane waves solutions, i.e. if one assumes that the metric coefficients are functions of the argument $u = (t - \lambda\,y)$ or of $v = (t + \lambda\,y)$ [14–17]. In the following we concentrate on solutions which depend on $(t - \lambda\,y)$:

$$a = a(t - \lambda\,y), \quad n = n(t - \lambda\,y), \quad \Phi = \Phi(t - \lambda\,y). \tag{22}$$

From the equation $G_y^t = 0$ we get

$$\frac{\overset{*}{a}}{n\Phi} = \text{constant}, \tag{23}$$

where the superscribed asterisk * denotes derivative with respect to u. Now considering the history of the three-brane as described by a point trajectory in the (t, y) spacetime it is possible to introduce a Gaussian normal coordinate system where $\Phi = 1$. Another ansatz where $n = 1$ is made in the literature [18,19] so we shall separately consider these two additional assumptions to solve the Einstein equations in the bulk.

Let us begin with the choice $n = 1$. One has from (23)

$$\Phi = \frac{\overset{*}{a}}{\beta}, \tag{24}$$

with β a constant. Now $G_t^t = G_y^y$ and the Einstein equations reduce to

$$\frac{a^2\,\overset{***}{a}}{\overset{*}{a}} + 4\,a\,\overset{**}{a} + \overset{*}{a}{}^2 = \Lambda_5\,a^2 + \beta^2\lambda^2, \tag{25a}$$

$$a \overset{**}{a} + \overset{*}{a}^2 = \frac{\Lambda_5}{3}\, a^2 + \beta^2 \lambda^2. \tag{25b}$$

Subtracting (25b) from (25a) we obtain an equation which is the first derivative of (25b) with respect to u, so both equations in (25) are satisfied solving

$$\overset{**}{(a^2)} = \frac{2}{3}\Lambda_5\, a^2 + 2\,\beta^2\lambda^2. \tag{26}$$

The solution is

$$a^2 = c_1 \sinh\sqrt{\frac{2}{3}\,\Lambda_5}\,(t-\lambda\, y) + c_2 \cosh\sqrt{\frac{2}{3}\,\Lambda_5}\,(t-\lambda\, y) - 3\,\frac{\beta^2\lambda^2}{\Lambda_5}, \tag{27}$$

with c_1 and c_2 suitable constants. Requiring $a^2(0) = 0$ and putting for future simplifications $c_1 = 0$ we have

$$a^2 = \frac{6\,\beta^2\lambda^2}{\Lambda_5}\, \sinh^2\sqrt{\frac{\Lambda_5}{6}}\,(t-\lambda\, y). \tag{28}$$

The other metric coefficient is

$$\Phi^2 = \lambda^2\, \cosh^2\sqrt{\frac{\Lambda_5}{6}}\,(t-\lambda\, y). \tag{29}$$

We notice that the choice $c_1 = 0$ implies $\overset{*}{(a^2)}(0) = 0$ and leaves the possibility of taking either sign for the 5D cosmological constant.

When $\Phi = 1$ one has from (23)

$$n = \frac{\overset{*}{a}}{\alpha}, \tag{30}$$

with α a constant. One has again $G^t_t = G^y_y$ and the Einstein equations reduce to

$$\frac{a^2\, \overset{***}{a}}{\overset{*}{a}} + 4\, a\, \overset{**}{a} + \overset{*}{a}^2 = -\frac{\Lambda_5}{\lambda^2}\, a^2 + \frac{\alpha^2}{\lambda^2}, \tag{31a}$$

$$a\, \overset{**}{a} + \overset{*}{a}^2 = -\frac{\Lambda_5}{3\,\lambda^2}\, a^2 + \frac{\alpha^2}{\lambda^2}. \tag{31b}$$

Subtracting (31b) from (31a) we obtain an equation which is the first derivative of (31b) with respect to u, so both equations in (25) are satisfied solving

$$\overset{**}{(a^2)} = -\frac{2\,\Lambda_5}{3\,\lambda^2}\, a^2 + 2\,\frac{\alpha^2}{\lambda^2}. \tag{32}$$

The solution is

$$a^2 = c_1 \sinh\frac{1}{\lambda}\sqrt{-\frac{2}{3}\,\Lambda_5}\,(t-\lambda\, y) + c_2 \cosh\frac{1}{\lambda}\sqrt{-\frac{2}{3}\,\Lambda_5}\,(t-\lambda\, y) - \frac{3\,\alpha^2}{\Lambda_5}. \tag{33}$$

Requiring $a^2(0) = 0$ and putting for future simplifications $c_1 = 0$ we have

$$a^2 = -\frac{6\,\alpha^2}{\Lambda_5}\, \sinh^2\frac{1}{\lambda}\sqrt{\frac{-\Lambda_5}{6}}\,(t-\lambda\, y). \tag{34}$$

The other metric coefficient is

$$n^2 = \frac{1}{\lambda^2}\, \cosh^2\frac{1}{\lambda}\sqrt{\frac{-\Lambda_5}{6}}\,(t-\lambda\, y). \tag{35}$$

Here again the choice $c_1 = 0$ implies $\overset{*}{(a^2)}(0) = 0$ and leaves the possibility of taking either sign for the 5D cosmological constant. When $\mathbf{Z_2}$ symmetry on reflection around the brane at $y = 0$ is assumed, the extra-coordinate y in the above solutions must be replaced by $|y|$.

5 Solutions in the brane

The 4D line element can be written as

$$\mathrm{d}s^2 = a_0^2(t)\,\mathrm{d}\sigma_k^2 - n_0^2(t)\,\mathrm{d}t^2, \tag{36}$$

where the subscript $_0$ here does not mean "calculated putting $y = 0$ in the corresponding bulk quantities" because those metric coefficients were found without taking into account the matter content of the brane, but means "calculated solving Einstein equations (11) on the brane with ${}^{(4)}G^\mu_\nu$ derived from the line element (36)". Before searching for plane wave solutions in the brane let us return, for completeness, to the case treated in Sect. 3 with 5D line element given in (15). The relevant Einstein equations obtained from (11) are

$${}^{(4)}G^r_r = {}^{(4)}G^\vartheta_\vartheta = {}^{(4)}G^\varphi_\varphi = -\frac{k}{a_0^2} - \frac{\dot a_0^2}{a_0^2 n_0^2} + \frac{2\,\dot a_0 \dot n_0}{a_0 n_0^3} - \frac{2\,\ddot a_0}{a_0 n_0^2} = -\frac{\Lambda_5}{2} - 7\,\widetilde{\kappa}^2 - \frac{k\,\mathrm{e}^{-2\,\widetilde{\chi}t}}{\widetilde{A}^2}, \tag{37a}$$

$${}^{(4)}G^t_t = -\frac{3\,k}{a_0^2} - \frac{3\,\dot a_0^2}{a_0^2 n_0^2} = -\frac{\Lambda_5}{2} - 7\,\widetilde{\kappa}^2 - \frac{3\,k\,\mathrm{e}^{-2\,\widetilde{\chi}t}}{\widetilde{A}^2}, \tag{37b}$$

where Λ_5 is given in (18).

The solutions are

$$a_0^2 = c_1\,\mathrm{e}^{2\,\widetilde{\chi}\,t} \equiv \widetilde{A}^2\,\mathrm{e}^{2\,\widetilde{\chi}\,t}, \tag{38}$$

$$n_0^2 = \frac{\dot a_0^2}{\frac{a_0^2}{3}\,(\Lambda_4 + \kappa_4^2\,\rho) - k} = 1, \tag{39}$$

where Λ_4 and ρ, written explicitly below, are respectively the cosmological constant and the matter density in the brane. With the choice $c_1 = \widetilde{A}^2$ the brane metric can be obtained evaluating the corresponding bulk metric at $y = 0$. Comparing (37) and (9) we have defined the 4D cosmological constant, including a possible contribution coming from the brane tension, as

$$\Lambda_4 = \frac{\Lambda_5}{2} + 7\,\widetilde{\kappa}^2 = \frac{4\,\kappa^2 + 3\,\chi^2}{1 - \frac{\chi^2}{\kappa^2}} \tag{40}$$

The remaining terms are the components of the effective energy-momentum tensor of the brane. This can be considered as the stress tensor of a perfect fluid at rest, because $G_{rt} = 0$ in the frame given by (36), with pressure p and density ρ given by

$$p = -\frac{1}{3}\,\rho, \quad \rho = \frac{3\,k}{\kappa_4^2\,a_0^2}. \tag{41}$$

Here and in the following we shall attribute to a "fluid" also quantities proportional to the curvature index k and to the cosmological constant Λ_4; moreover all the fluids taken into account may cause violations of some of the energy conditions. Comparing the values of the Hubble constant obtained from its definition and from the Friedmann equation (37b),

$$H^2 = \left(\frac{\dot a_0}{a_0 n_0}\right)^2 = \widetilde{\chi}^2 = \frac{\Lambda_4}{3} - \frac{k}{a_0^2} + \frac{\kappa_4^2}{3}\,\rho = \frac{\Lambda_4}{3}, \tag{42}$$

we obtain $\widetilde{\chi} = H = \sqrt{\Lambda_4/3}$ for any value of the curvature index k so $a_0 \propto \mathrm{e}^{H\,t}$ as in the de Sitter model, t now being the proper time in the brane. Clearly the deceleration parameter $q = -(a_0\,\ddot a_0)/\dot a^2$ and the density parameter $\Omega_\Lambda = \Lambda_4/(3\,H^2)$ have the values $q = -1$ and $\Omega_\Lambda = 1$.

The next case we shall treat corresponds to the choice $n = 1$ in the bulk. The relevant Einstein equations now are

$$-\frac{k}{a_0^2} - \frac{\dot a_0^2}{a_0^2 n_0^2} + \frac{2\,\dot a_0 \dot n_0}{a_0 n_0^3} - \frac{2\,\ddot a_0}{a_0 n_0^2} = -\frac{\Lambda_5}{2} - \frac{2\,\Lambda_5}{3\,\sinh^2\sqrt{\frac{\Lambda_5}{6}}\,t} - \frac{k}{\frac{6\,\beta^2\lambda^2}{\Lambda_5}\,\sinh^2\sqrt{\frac{\Lambda_5}{6}}\,t}, \tag{43a}$$

$$-\frac{3\,k}{a_0^2} - \frac{3\,\dot a_0^2}{a_0^2 n_0^2} = -\frac{\Lambda_5}{2} - \frac{\Lambda_5}{\sinh^2\sqrt{\frac{\Lambda_5}{6}}\,t} - \frac{3\,k}{\frac{6\,\beta^2\lambda^2}{\Lambda_5}\,\sinh^2\sqrt{\frac{\Lambda_5}{6}}\,t}. \tag{43b}$$

The solutions to the system (43) are

$$a_0^2 = c_1\left(\sinh^2\sqrt{\frac{\Lambda_4}{3}}t\right)^{2-k/(k+\beta^2\lambda^2)}, \tag{44}$$

$$n_0^2 = \frac{\dot a_0^2}{\frac{a_0^2}{3}\,\left(\Lambda_4 + \kappa_4^2\,(\rho_1+\rho_2)\right) - k}. \tag{45}$$

Comparing (43) and (9), we have defined the 4D cosmological constant as $\Lambda_4 = \frac{\Lambda_5}{2}$ and have considered the remaining terms as the superposition of two different fluids with pressures and densities given by

$$p_1 = -\frac{2}{3}\,\rho_1, \qquad \rho_1 = \frac{2\,\Lambda_4}{\kappa_4^2\,\sinh^2\sqrt{\frac{\Lambda_4}{3}}\,t} \propto a_0^{-2\,(k+\beta^2\lambda^2)/(k+2\,\beta^2\lambda^2)}, \tag{46}$$

$$p_2 = -\frac{1}{3}\,\rho_2, \qquad \rho_2 = \frac{k}{\frac{\beta^2\lambda^2\kappa_4^2}{\Lambda_4}\,\sinh^2\sqrt{\frac{\Lambda_4}{3}}\,t} \propto a_0^{-2\,(k+\beta^2\lambda^2)/(k+2\,\beta^2\lambda^2)}. \tag{47}$$

From (43b) we obtain the Friedmann equation

$$H^2 = \left(\frac{\dot a_0}{a_0 n_0}\right)^2 = \frac{\Lambda_4}{3} - \frac{k}{a_0^2} + \frac{\kappa_4^2}{3}\,(\rho_1+\rho_2). \tag{48}$$

Here the density $\rho = \rho_1 + \rho_2$ appears linearly as in the usual Friedmann equation. This linear dependence is common to other induced gravity models in the literature [20] and appears, of course with different values of the quantities involved, in all the brane solutions found in this paper. Also we notice that in the actual case it is neither possible to recover the brane metric evaluating the bulk metric at $y = 0$ nor in general to give the explicit dependence of the scale factor on the proper time τ. As a qualitative estimate one can say that a_0^2, starting from zero, either increases indefinitely or oscillates depending on the values of the parameters which appear in (44). Exact results can however be obtained if we assume $k = 0$, which is a value strongly suggested for our universe at the present time. For a spatially flat universe we have from (48)

$$a_0 \propto \sinh^2\sqrt{\frac{\Lambda_4}{12}}\,\tau, \qquad H^2 = \frac{\Lambda_4}{3}\,\coth^2\sqrt{\frac{\Lambda_4}{12}}\,\tau$$

and

$$q = -\frac{1}{2}\,\left(1 + \frac{\Lambda_4}{3H^2}\right) \tag{49}$$

Recalling that the density parameter $\Omega_\Lambda = \Lambda_4/(3\,H^2)$ varies in the interval $[0,1]$, one obtains for the deceleration parameter $-1 \le q \le -\frac{1}{2}$ and for the age of the universe $\tau_0 = 2/H_0\,\left(\sqrt{1/(\Omega_\Lambda)_0}\,\coth^{-1}\sqrt{1/(\Omega_\Lambda)_0}\right)$ where the subscript $_0$ here refers to present values. While the range of values for q is in agreement with the observational results, this does not happen for the age of the universe whose minimum value here as a result is equal to $2/H_0$.

Let us finally treat the case which corresponds to the choice $\phi = 1$ in the bulk. The relevant Einstein equations are

$$-\frac{k}{a_0^2} - \frac{\dot a_0^2}{a_0^2 n_0^2} + \frac{2\,\dot a_0 \dot n_0}{a_0 n_0^3} - \frac{2\,\ddot a_0}{a_0 n_0^2} = \frac{2\,\Lambda_5}{3} - \frac{\Lambda_5}{6}\left(\frac{4}{\sin^2\frac{1}{\lambda}\sqrt{\frac{\Lambda_5}{6}}\,t} + \frac{1}{\cos^2\frac{1}{\lambda}\sqrt{\frac{\Lambda_5}{6}}\,t}\right) - \frac{k}{\frac{6\,\alpha^2}{\Lambda_5}\,\sin^2\frac{1}{\lambda}\sqrt{\frac{\Lambda_5}{6}}\,t}, \tag{50a}$$

$$-\frac{3\,k}{a_0^2} - \frac{3\,\dot a_0^2}{a_0^2 n_0^2} = \frac{2\,\Lambda_5}{3} - \frac{\Lambda_5}{6}\left(\frac{6}{\sin^2\frac{1}{\lambda}\sqrt{\frac{\Lambda_5}{6}}\,t} + \frac{1}{\cos^2\frac{1}{\lambda}\sqrt{\frac{\Lambda_5}{6}}\,t}\right)$$

$$-\frac{3\,k}{\frac{6\,\alpha^2}{\Lambda_5}\,\sin^2\frac{1}{\lambda}\sqrt{\frac{\Lambda_5}{6}}\,t}. \tag{50b}$$

The solutions to the system (50) are

$$a_0^2 = c_1\,\exp\left(-\frac{\alpha^2}{3\,(k+\alpha^2)\,\cos^2\frac{1}{\lambda}\sqrt{\frac{-\Lambda_4}{4}}\,t}\right) \times\frac{\left(\sin^2\frac{1}{\lambda}\sqrt{\frac{-\Lambda_4}{4}}\,t\right)^{2-k/(k+\alpha^2)}}{\left(\cos^2\frac{1}{\lambda}\sqrt{\frac{-\Lambda_4}{4}}\,t\right)^{\alpha^2/(3\,(k+\alpha^2))}}, \tag{51}$$

$$n_0^2 = \frac{\dot{a}_0^2}{\frac{a_0^2}{3}\,\left(\Lambda_4+\kappa_4^2\,(\rho_1+\rho_2+\rho_3)\right)-k}. \tag{52}$$

Comparing (43) and (9) we have defined the 4D cosmological constant as $\Lambda_4 = -\frac{2\,\Lambda_5}{3}$ and have considered the remaining terms as the superposition of three different fluids with pressures and densities given by

$$p_1 = -\frac{2}{3}\,\rho_1, \quad \rho_1 = -\frac{3\,\Lambda_4}{2\,\kappa_4^2\,\sin^2\frac{1}{\lambda}\sqrt{\frac{-\Lambda_4}{4}}\,t}, \tag{53}$$

$$p_2 = -\rho_2, \quad \rho_2 = -\frac{\Lambda_4}{4\,\kappa_4^2\,\cos^2\frac{1}{\lambda}\sqrt{\frac{-\Lambda_4}{4}}\,t}, \tag{54}$$

$$p_3 = -\frac{1}{3}\,\rho_3, \quad \rho_3 = -\frac{3\,k}{\frac{4\,\alpha^2}{\Lambda_4}\,\kappa_4^2\,\sin^2\frac{1}{\lambda}\sqrt{\frac{-\Lambda_4}{4}}\,t}. \tag{55}$$

Again we notice that in the actual case it is neither possible to recover the brane metric evaluating the bulk metric at $y = 0$ nor in general to give the explicit dependence of the scale factor on the proper time τ even assuming $k = 0$. As a qualitative estimate one can say that the scale factor a_0^2, starting from zero, either increases indefinitely or, after increasing, reaches again the zero depending on the values of the parameters appearing in (51).

Finally we notice that all the brane solutions given in this section satisfy the conservation equation of the brane energy-momentum tensor

$$\dot{\rho} + 3\,(\rho+p)\,\frac{\dot{a}_0}{a_0} = 0, \tag{56}$$

where ρ and p are, respectively, the sum of the densities and pressures of all the fluids which constitute the energy-momentum tensor in the brane Einstein equations.

6 Conclusions

We have studied brane-world cosmologies where the Einstein-Hilbert action is modified by a 4D scalar curvature from induced gravity on the brane. To investigate cosmological solutions in the brane–bulk system, we extended the static Randall–Sundrum line element to a simple dynamical case and from the knowledge of the metric so obtained we solved the 4D Einstein equations on the brane. The general features of the model are presented and the evolution of physically meaningfull quantities can be determined choosing suitably the model parameters. In our description the cosmological fluid appears as a mixture of perfect fluids obeying simple equations of state with constant barotropic factors. Of course our model has to be implemented to can describe the distinct periods of the universe evolution, a task which will require to consider more complicated equations of state for real fluids. If one adopts an approach of this kind, the simple model described here may serve as a basis for obtaining more detailed braneworld solutions and, therefore, a better comparison with the accumulated cosmological observations.

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